



# On Hyers–Ulam stability of two functional equations in non-Archimedean spaces

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**Abstract.** In this paper we prove, using the fixed point method, the generalized Hyers–Ulam stability of two functional equations in complete non-Archimedean normed spaces. One of these equations characterizes multi-Cauchy–Jensen mappings, and the other gives a characterization of multi-additive-quadratic mappings.

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## 1. Introduction

Throughout this paper,  $\mathbb{N}$  stands for the set of all positive integers,  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $\mathbb{R}_+ := [0, \infty)$ ,  $n \in \mathbb{N}$  and  $k \in \{0, \dots, n\}$ . Moreover, given a nonempty set  $V$ , we identify  $x = (x_1, \dots, x_n) \in V^n$  with  $(x^1, x^2) \in V^k \times V^{n-k}$ , where  $x^1 := (x_1, \dots, x_k)$  and  $x^2 := (x_{k+1}, \dots, x_n)$ .

It is well known that among functional equations, the *Cauchy equation*

$$f(x+y) = f(x) + f(y),$$

the *Jensen equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

(which is closely connected with the notion of a convex function) and the *Jordan–von Neumann (quadratic) equation*

$$q(x+y) + q(x-y) = 2q(x) + 2q(y)$$

(which is useful in some characterizations of inner product spaces) play a prominent role. A lot of information about them (in particular, about their solutions (which are said to be *additive*, *Jensen* and *quadratic mappings*,

respectively) and stability) and their applications can be found, for instance, in [18, 20, 25].

Let us recall that a semigroup  $G$  is called *uniquely divisible by 2* provided that for every  $x \in G$  there exists a unique  $y \in G$  (which is denoted by  $\frac{x}{2}$  or  $\frac{1}{2}x$ ) such that  $x = 2y$ . Given two semigroups  $G$  and  $H$  which are uniquely divisible by 2, we say (see also [8]) that a function  $f : G^n \rightarrow H$  is *k-Cauchy and n - k-Jensen* (briefly, *multi-Cauchy-Jensen*) if  $f$  satisfies Cauchy's functional equation in each of some  $k$  variables and Jensen's functional equation in each of the other variables. Let us note that for  $k = n$  the above definition leads to the so-called *multi-additive mappings* (some basic facts on such mappings can be found, for instance, in [20], where their application to the representation of polynomial functions is also presented); for  $k = 0$  we obtain the notion of *multi-Jensen function* (which was introduced in 2005 by Prager and Schwaiger (see [24]) with the connection with generalized polynomials); a 1-Cauchy and 1-Jensen mapping is just a *Cauchy-Jensen mapping* defined by Park and Bae in [22].

Similarly, a function  $f : V^n \rightarrow W$ , where  $V$  is a commutative group and  $W$  is a linear space, is called *k-additive and n - k-quadratic* (briefly, *multi-additive-quadratic*) if  $f$  is additive in each of some  $k$  variables and is quadratic in each of the other variables. For  $k = n$  the above definition leads to the multi-additive mappings; for  $k = 0$  we obtain the notion of *multi-quadratic function* (see [11]), and a 1-additive and 1-quadratic mapping is just an *additive-quadratic mapping* defined by Park, Bae and Chung in [23].

In [1, 2], the authors reduced the systems of  $n$  equations defining the multi-Cauchy-Jensen and multi-additive-quadratic mappings to single functional equations, and proved the stability of these equations in Banach spaces.

In this paper, we show their generalized (in the spirit of Bourgin [4] and Găvruta [15]) Hyers-Ulam stability in complete non-Archimedean normed spaces. Our results are significant supplements and/or generalizations of some results from [12, 14, 21, 26, 27].

Let us recall that, speaking of the stability of a functional equation, we follow the question raised in 1940 by Ulam: "*When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?*". The first partial answer (in the case of Cauchy's equation in Banach spaces) to Ulam's question was given by Hyers. After his result, a great number of papers (see, for instance, [5, 6, 9, 10, 13, 16, 17, 18] and the references therein) on the subject has been published, generalizing Ulam's problem and Hyers' theorem in various directions and to other (not only functional) equations.

In the proofs of our main results (Theorems 2.3 and 3.2), we use the fixed point method, which was used for the investigation of the Hyers-Ulam stability of functional equations for the first time by Baker in [3]. For more information about this method we refer the reader to the recent survey papers [5, 6, 13].

Let us now recall (see, for instance, [19]) some basic definitions and facts concerning non-Archimedean normed spaces.

By a *non-Archimedean field* we mean a field  $\mathbb{K}$  equipped with a function (called *valuation*)  $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}_+$  such that

$$\begin{aligned} |r| &= 0 && \text{if and only if } r = 0, \\ |rs| &= |r| |s|, && r, s \in \mathbb{K}, \\ |r + s| &\leq \max\{|r|, |s|\}, && r, s \in \mathbb{K}. \end{aligned}$$

In any non-Archimedean field we have  $|1| = |-1| = 1$  and  $|n| \leq 1$  for  $n \in \mathbb{N}_0$ .

In any field  $\mathbb{K}$  the function  $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}_+$  given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are *p-adic numbers* which have gained the interest of physicists for their research in some problems coming from quantum physics, *p*-adic strings and superstrings.

Let  $X$  be a linear space over a field  $\mathbb{K}$  with a non-Archimedean nontrivial valuation  $|\cdot|$ . A function  $\|\cdot\| : X \rightarrow \mathbb{R}_+$  is said to be a *non-Archimedean norm* if it satisfies the following conditions:

$$\begin{aligned} \|x\| &= 0 && \text{if and only if } x = 0, \\ \|rx\| &= |r| \|x\|, && r \in \mathbb{K}, x \in X, \\ \|x + y\| &\leq \max\{\|x\|, \|y\|\}, && x, y \in X. \end{aligned}$$

Then  $(X, \|\cdot\|)$  is called a *non-Archimedean normed space*.

In any such a space, the function  $d : X \times X \rightarrow \mathbb{R}_+$ , given by

$$d(x, y) = \|x - y\|, \quad x, y \in X,$$

is a metric on  $X$ . Recall also that a sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of a non-Archimedean normed space is Cauchy if and only if  $(x_{n+1} - x_n)_{n \in \mathbb{N}}$  converges to zero. Moreover, the addition, scalar multiplication and non-Archimedean norm are continuous mappings.

The first work on the Hyers–Ulam stability of functional equations in complete non-Archimedean normed spaces (some particular cases were considered earlier; see [5] for details) is [21]. After it, a lot of papers (see, for instance, [14, 27] and the references therein) on the stability of other equations in such spaces have been published.

## 2. Stability of an equation characterizing multi-Cauchy–Jensen mappings

In [2], the following characterization of multi-Cauchy–Jensen mappings was proved.

**Theorem 2.1.** *Assume that  $V$  is a semigroup uniquely divisible by 2 and with an identity element, and  $W$  is a linear space over the rationals. Then a*

function  $f : V^n \rightarrow W$  is  $k$ -Cauchy and  $n - k$ -Jensen if and only if for any  $(x_i^1, x_i^2) = (x_{1i}, \dots, x_{ni}) \in V^n$ ,  $i \in \{1, 2\}$ , we have

$$2^{n-k} f\left(x_1^1 + x_2^1, \frac{x_1^2 + x_2^2}{2}\right) = \sum_{i_1, \dots, i_n \in \{1, 2\}} f(x_{1i_1}, \dots, x_{ni_n}). \quad (2.1)$$

In this section, we show the generalized Hyers–Ulam stability of equation (2.1) in complete non-Archimedean normed spaces (its stability in Banach spaces was proved in [2]). The proof is based on a fixed point result that can be derived from [7, Theorem 1]. To present it, we introduce the following three hypotheses:

- (H1)  $E$  is a nonempty set,  $Y$  is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2,  $j \in \mathbb{N}$ ,  $f_1, \dots, f_j : E \rightarrow E$  and  $L_1, \dots, L_j : E \rightarrow \mathbb{R}_+$ ;  
 (H2)  $\mathcal{T} : Y^E \rightarrow Y^E$  is an operator satisfying the inequality

$$\begin{aligned} & \|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \\ & \leq \max_{i \in \{1, \dots, j\}} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in Y^E, x \in E; \end{aligned}$$

- (H3)  $\Lambda : \mathbb{R}_+^E \rightarrow \mathbb{R}_+^E$  is an operator defined by

$$\Lambda\delta(x) := \max_{i \in \{1, \dots, j\}} L_i(x) \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^E, x \in E.$$

Now, we are in a position to present the mentioned fixed point theorem.

**Theorem 2.2.** *Let hypotheses (H1)–(H3) hold and the functions  $\varepsilon : E \rightarrow \mathbb{R}_+$  and  $\varphi : E \rightarrow Y$  fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in E,$$

and

$$\lim_{l \rightarrow \infty} \Lambda^l \varepsilon(x) = 0, \quad x \in E.$$

Then for every  $x \in E$ , the limit

$$\lim_{l \rightarrow \infty} \mathcal{T}^l \varphi(x) =: \psi(x)$$

exists and the function  $\psi \in Y^E$ , defined in this way, is a fixed point of  $\mathcal{T}$  with

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l \in \mathbb{N}_0} \Lambda^l \varepsilon(x), \quad x \in E.$$

Given  $f : V^n \rightarrow W$ ,  $x_i^1 := (x_{1i}, \dots, x_{ki}) \in V^k$ ,  $x_i^2 := (x_{k+1i}, \dots, x_{ni}) \in V^{n-k}$ ,  $i \in \{1, 2\}$ , put

$$\begin{aligned} & \Phi(f)(x_1^1, x_2^1, x_1^2, x_2^2) \\ & := 2^{n-k} f\left(x_1^1 + x_2^1, \frac{x_1^2 + x_2^2}{2}\right) - \sum_{i_1, \dots, i_n \in \{1, 2\}} f(x_{1i_1}, \dots, x_{ni_n}). \end{aligned}$$

With this notation, we have the following result.

**Theorem 2.3.** *Suppose that  $V$  is a linear space over the rationals and let  $W$  be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. Let  $f : V^n \rightarrow W$  and  $\theta : V^n \times V^n \rightarrow \mathbb{R}_+$  be mappings satisfying the inequality*

$$\|\Phi(f)(x_1^1, x_2^1, x_1^2, x_2^2)\| \leq \theta(x_1^1, x_2^1, x_1^2, x_2^2) \quad (2.2)$$

for  $x_1^1, x_2^1 \in V^k, x_1^2, x_2^2 \in V^{n-k}$ . Assume also that there is an  $s \in \{-1, 1\}$  such that

$$\lim_{l \rightarrow \infty} \left( \frac{1}{|2|^{sk}} \right)^l \theta(2^{sl} x_1^1, x_1^2, 2^{sl} x_2^1, x_2^2) = 0 \quad (2.3)$$

for  $x_1^1, x_2^1 \in V^k, x_1^2, x_2^2 \in V^{n-k}$ . Then there exists a solution  $F : V^n \rightarrow W$  of equation (2.1) with

$$\|f(x) - F(x)\| \leq \sup_{l \in \mathbb{N}_0} \frac{1}{|2|^{n+k \frac{s-1}{2}}} \left( \frac{1}{|2|^{sk}} \right)^l \theta \left( 2^{sl + \frac{s-1}{2}} x^1, x^2, 2^{sl + \frac{s-1}{2}} x^1, x^2 \right) \quad (2.4)$$

for  $x = (x^1, x^2) \in V^n$ .

*Proof.* Putting  $x_1^1 = x_2^1 = x^1 \in V^k$  and  $x_1^2 = x_2^2 = x^2 \in V^{n-k}$  in (2.2), we get

$$\|2^{n-k} f(2x^1, x^2) - 2^n f(x)\| \leq \theta(x, x), \quad x \in V^n,$$

whence

$$\left\| \frac{1}{2^k} f(2x^1, x^2) - f(x) \right\| \leq \frac{1}{|2|^n} \theta(x, x), \quad x \in V^n. \quad (2.5)$$

Similarly,

$$\left\| 2^k f \left( \frac{1}{2} x^1, x^2 \right) - f(x) \right\| \leq \frac{1}{|2|^{n-k}} \theta \left( \frac{1}{2} x^1, x^2, \frac{1}{2} x^1, x^2 \right), \quad x \in V^n. \quad (2.6)$$

Fix an  $x \in V^n$  and write

$$\begin{aligned} \mathcal{T}\xi(x) &:= \frac{1}{2^{sk}} \xi(2^s x^1, x^2), \quad \xi \in W^{V^n}, \\ \varepsilon(x) &:= \begin{cases} \frac{1}{|2|^n} \theta(x, x) & \text{if } s = 1, \\ \frac{1}{|2|^{n-k}} \theta \left( \frac{1}{2} x^1, x^2, \frac{1}{2} x^1, x^2 \right) & \text{if } s = -1. \end{cases} \end{aligned}$$

Then, by (2.5) and (2.6), we obtain

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in V^n.$$

Next, put

$$\Lambda\eta(x) := \frac{1}{|2|^{sk}} \eta(2^s x^1, x^2), \quad \eta \in \mathbb{R}_+^{V^n}, \quad x \in V^n.$$

It is easily seen that  $\Lambda$  has the form described in (H3) with  $E = V^n, j = 1$  and  $f_1(x) = (2^s x^1, x^2), L_1(x) = \frac{1}{|2|^{sk}}$  for  $x \in V^n$ . Moreover, for any  $\xi, \mu \in W^{V^n}$

and  $x \in V^n$  we have

$$\begin{aligned} \|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| &= \left\| \frac{1}{2^{sk}} \xi(2^s x^1, x^2) - \frac{1}{2^{sk}} \mu(2^s x^1, x^2) \right\| \\ &\leq L_1(x) \|\xi(f_1(x)) - \mu(f_1(x))\|, \end{aligned}$$

so hypothesis (H2) is also valid.

Finally, using induction, one can check that for any  $l \in \mathbb{N}_0$  and  $x \in V^n$  we have

$$\begin{aligned} \Lambda^l \varepsilon(x) &= \left( \frac{1}{|2|^{sk}} \right)^l \varepsilon(2^{sl} x^1, x^2) \\ &= \frac{1}{|2|^{n+k\frac{s-1}{2}}} \left( \frac{1}{|2|^{sk}} \right)^l \theta \left( 2^{sl+\frac{s-1}{2}} x^1, x^2, 2^{sl+\frac{s-1}{2}} x^1, x^2 \right), \end{aligned}$$

which, together with (2.3), shows that all assumptions of Theorem 2.2 are satisfied. Therefore, there exists a function  $F : V^n \rightarrow W$  such that

$$F(x) = \frac{1}{2^{sk}} F(2^s x^1, x^2), \quad x \in V^n,$$

and (2.4) holds. Moreover,

$$F(x) = \lim_{l \rightarrow \infty} \mathcal{T}^l f(x), \quad x \in V^n.$$

One can now show, by induction, that

$$\|\Phi(\mathcal{T}^l f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \leq \left( \frac{1}{|2|^{sk}} \right)^l \theta(2^{sl} x_1^1, x_1^2, 2^{sl} x_2^1, x_2^2) \quad (2.7)$$

for  $l \in \mathbb{N}_0$ ,  $x_1^1, x_2^1 \in V^k$  and  $x_1^2, x_2^2 \in V^{n-k}$ . Letting  $l \rightarrow \infty$  in (2.7) and using (2.3), we obtain

$$\Phi(F)(x_1^1, x_1^2, x_2^1, x_2^2) = 0,$$

which means that the function  $F$  satisfies equation (2.1).  $\square$

Theorem 2.3 with  $\theta \equiv \varepsilon > 0$  and  $s = -1$  yields immediately the following result.

**Corollary 2.4.** *Assume that  $k \geq 1$ ,  $\varepsilon > 0$ ,  $V$  is a linear space over the rationals and  $W$  is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2 such that  $|2| < 1$ . If  $f : V^n \rightarrow W$  satisfies the inequality*

$$\|\Phi(f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \leq \varepsilon$$

*for  $x_1^1, x_2^1 \in V^k$ ,  $x_1^2, x_2^2 \in V^{n-k}$ , then there exists a solution  $F : V^n \rightarrow W$  of equation (2.1) with*

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{|2|^{n-k}}, \quad x \in V^n.$$

### 3. Stability of an equation characterizing multi-additive-quadratic mappings

In this section, we use Theorem 2.2 to prove the generalized Hyers–Ulam stability of an equation characterizing multi-additive-quadratic mappings in complete non-Archimedean normed spaces (its stability in Banach spaces was shown in [1]).

Before we recall this characterization (obtained in [1]) and state the stability result, let us introduce an additional notation. Namely, given an  $m \in \mathbb{N}$ , for any  $p = (p_1, \dots, p_m), q = (q_1, \dots, q_m) \in \{-2, -1, 0, 1, 2\}^m$ ,  $l \in \mathbb{N}_0$ , we put  $lp := (lp_1, \dots, lp_m)$  and  $pq := (p_1q_1, \dots, p_mq_m)$ .

**Theorem 3.1.** *Let  $V$  be a commutative group and  $W$  a linear space over the rationals. Then a function  $f : V^n \rightarrow W$  is  $k$ -additive and  $n - k$ -quadratic if and only if for any  $(x_i^1, x_i^2) = (x_{1i}, \dots, x_{ni}) \in V^n$ ,  $i \in \{1, 2\}$ , we have*

$$\sum_{q \in \{-1, 1\}^{n-k}} f(x_1^1 + x_2^1, x_1^2 + qx_2^2) = 2^{n-k} \sum_{i_1, \dots, i_n \in \{1, 2\}} f(x_{1i_1}, \dots, x_{ni_n}). \quad (3.1)$$

In what follows,  $P$  stands for  $\{-1, 1\}^{n-k}$ . Moreover, given  $f : V^n \rightarrow W$ ,  $x_i^1 \in V^k$ ,  $x_i^2 \in V^{n-k}$ ,  $i \in \{1, 2\}$ , put

$$\begin{aligned} \Psi(f)(x_1^1, x_1^2, x_2^1, x_2^2) \\ := \sum_{q \in P} f(x_1^1 + x_2^1, x_1^2 + qx_2^2) - 2^{n-k} \sum_{i_1, \dots, i_n \in \{1, 2\}} f(x_{1i_1}, \dots, x_{ni_n}). \end{aligned}$$

With this notation, we have the following result.

**Theorem 3.2.** *Let  $V$  be a commutative group,  $W$  a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2,  $f : V^n \rightarrow W$ ,  $\theta : V^n \times V^n \rightarrow \mathbb{R}_+$ . Assume also that for any  $x_1^1, x_2^1 \in V^k$ ,  $x_1^2, x_2^2 \in V^{n-k}$ ,*

$$\|\Psi(f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \leq \theta(x_1^1, x_1^2, x_2^1, x_2^2) \quad (3.2)$$

and

$$\lim_{l \rightarrow \infty} \left( \frac{1}{|2|^{2n-k}} \right)^l \max_{p \in P} \theta(2^l(x_1^1, px_1^2), 2^l(x_2^1, px_2^2)) = 0. \quad (3.3)$$

Then there exists a solution  $F : V^n \rightarrow W$  of equation (3.1) with

$$\|f(x) - F(x)\| \leq \sup_{l \in \mathbb{N}_0} \left( \frac{1}{|2|^{2n-k}} \right)^{l+1} \max_{p \in P} \theta(2^l(x^1, px^2), 2^l(x^1, px^2)) \quad (3.4)$$

for  $x = (x^1, x^2) \in V^n$ .

*Proof.* Putting  $x_1^1 = x_2^1 = x^1 \in V^k$  and  $x_1^2 = x_2^2 = x^2 \in V^{n-k}$  in (3.2), we get

$$\left\| \sum_{p \in P} f(2x^1, 2px^2) - 2^{n-k} 2^n f(x) \right\| \leq \theta(x, x), \quad x \in V^n,$$

whence

$$\left\| \frac{1}{2^{2n-k}} \sum_{p \in P} f(2x^1, 2px^2) - f(x) \right\| \leq \frac{1}{|2|^{2n-k}} \theta(x, x), \quad x \in V^n. \quad (3.5)$$

Fix an  $x \in V^n$  and write

$$\begin{aligned} \mathcal{T}\xi(x) &:= \frac{1}{2^{2n-k}} \sum_{p \in P} \xi(2x^1, 2px^2), \quad \xi \in W^{V^n}, \\ \varepsilon(x) &:= \frac{1}{|2|^{2n-k}} \theta(x, x). \end{aligned}$$

Then, by (3.5), we obtain

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in V^n.$$

Next, put

$$\Lambda\eta(x) := \max_{p \in P} \frac{1}{|2|^{2n-k}} \eta(2x^1, 2px^2), \quad \eta \in \mathbb{R}_+^{V^n}, \quad x \in V^n.$$

It is easily seen that  $\Lambda$  has the form described in (H3). Moreover, for any  $\xi, \mu \in W^{V^n}$  and  $x \in V^n$  we have

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \max_{p \in P} \frac{1}{|2|^{2n-k}} \|\xi(2x^1, 2px^2) - \mu(2x^1, 2px^2)\|,$$

so hypothesis (H2) is also valid.

Finally, using induction, one can check that for any  $l \in \mathbb{N}$  and  $x \in V^n$  we have

$$\Lambda^l \varepsilon(x) = \left( \frac{1}{|2|^{2n-k}} \right)^l \max_{p \in P} \varepsilon(2^l x^1, 2^l p x^2),$$

which, together with (3.3), shows that all assumptions of Theorem 2.2 are satisfied. Therefore, there exists a function  $F : V^n \rightarrow W$  such that

$$F(x) = \frac{1}{2^{2n-k}} \sum_{p \in P} F(2x^1, 2px^2), \quad x \in V^n,$$

and (3.4) holds. Moreover,

$$F(x) = \lim_{l \rightarrow \infty} \mathcal{T}^l f(x), \quad x \in V^n.$$

Now, we show that

$$\begin{aligned} &\|\Psi(\mathcal{T}^l f)(x_1^1, x_1^2, x_2^2, x_2^2)\| \\ &\leq \left( \frac{1}{|2|^{2n-k}} \right)^l \max_{p \in P} \theta(2^l(x_1^1, p x_1^2), 2^l(x_2^1, p x_2^2)) \end{aligned} \quad (3.6)$$



for  $l \in \mathbb{N}$ ,  $x_1^1, x_2^1 \in V^k$  and  $x_1^2, x_2^2 \in V^{n-k}$ . In order to do this, fix  $x_1^1, x_2^1 \in V^k$ ,  $x_1^2, x_2^2 \in V^{n-k}$ . If  $l = 1$ , then

$$\begin{aligned}
 & \|\Psi(\mathcal{T}f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \\
 &= \left\| \sum_{q \in P} \mathcal{T}f(x_1^1 + x_2^1, x_1^2 + qx_2^2) - 2^{n-k} \sum_{i_1, \dots, i_n \in \{1, 2\}} \mathcal{T}f(x_{1i_1}, \dots, x_{ni_n}) \right\| \\
 &= \left\| \sum_{q \in P} \frac{1}{2^{2n-k}} \sum_{p \in P} f(2(x_1^1 + x_2^1), 2px_1^1 + 2pqx_2^1) \right. \\
 &\quad \left. - \frac{2^{n-k}}{2^{2n-k}} \sum_{i_1, \dots, i_n \in \{1, 2\}} \sum_{p \in P} f(2x_{1i_1}, \dots, 2x_{ki_k}, p(2x_{k+1i_{k+1}}, \dots, 2x_{ni_n})) \right\| \\
 &= \left\| \frac{1}{2^{2n-k}} \sum_{p \in P} \Psi(f)(2x_1^1, 2px_1^2, 2x_2^1, 2px_2^2) \right\| \\
 &\leq \frac{1}{2^{|2n-k|}} \max_{p \in P} \|\Psi(f)(2(x_1^1, px_1^2), 2(x_2^1, px_2^2))\| \\
 &\leq \frac{1}{2^{|2n-k|}} \max_{p \in P} \theta(2(x_1^1, px_1^2), 2(x_2^1, px_2^2)).
 \end{aligned}$$

Next, assume that (3.6) holds for an  $l \in \mathbb{N}$ . Then

$$\begin{aligned}
 & \|\Psi(\mathcal{T}^{l+1}f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \\
 &= \left\| \sum_{q \in P} \mathcal{T}^{l+1}f(x_1^1 + x_2^1, x_1^2 + qx_2^2) \right. \\
 &\quad \left. - 2^{n-k} \sum_{i_1, \dots, i_n \in \{1, 2\}} \mathcal{T}^{l+1}f(x_{1i_1}, \dots, x_{ni_n}) \right\| \\
 &= \left\| \sum_{q \in P} \frac{1}{2^{2n-k}} \sum_{p \in P} \mathcal{T}^l f(2(x_1^1 + x_2^1), 2px_1^1 + 2pqx_2^1) - \frac{2^{n-k}}{2^{2n-k}} \right. \\
 &\quad \left. \times \sum_{i_1, \dots, i_n \in \{1, 2\}} \sum_{p \in P} \mathcal{T}^l f(2x_{1i_1}, \dots, 2x_{ki_k}, p(2x_{k+1i_{k+1}}, \dots, 2x_{ni_n})) \right\| \\
 &= \left\| \frac{1}{2^{2n-k}} \sum_{p \in P} \Psi(\mathcal{T}^l f)(2x_1^k, 2px_1^2, 2x_2^1, 2px_2^2) \right\| \\
 &\leq \left( \frac{1}{2^{|2n-k|}} \right)^{l+1} \max_{p \in P} \theta(2^{l+1}(x_1^1, px_1^2), 2^{l+1}(x_2^1, px_2^2)).
 \end{aligned}$$

Letting  $l \rightarrow \infty$  in (3.6) and using (3.3), we obtain

$$\Psi(F)(x_1^1, x_1^2, x_2^1, x_2^2) = 0,$$

which means that the function  $F$  satisfies equation (3.1).  $\square$

If

$$\theta(x_1^1, x_1^2, x_2^1, x_2^2) = \sum_{i=1}^2 \sum_{j=1}^n \|x_{ji}\|^t,$$

then for any  $p \in P$ ,  $l \in \mathbb{N}_0$ ,  $x_1^1, x_1^2 \in V^k$ ,  $x_2^1, x_2^2 \in V^{n-k}$ , the equality

$$\theta(2^l(x^1, px^2), 2^l(x^1, px^2)) = |2|^{lt} \theta((x^1, x^2), (x^1, x^2))$$

holds. Moreover, (3.3) is fulfilled under the additional assumptions  $|2| < 1$  and  $t > 2n - k$ . Therefore, Theorem 3.2 yields the following result.

**Corollary 3.3.** *Assume that  $t \in \mathbb{R}$  fulfills  $t > 2n - k$ ,  $V$  is a normed space and  $W$  is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2 such that  $|2| < 1$ . If  $f : V^n \rightarrow W$  satisfies the inequality*

$$\|\Psi(f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \leq \sum_{i=1}^2 \sum_{j=1}^n \|x_{ji}\|^t$$

for  $x_1^1, x_1^2 \in V^k$ ,  $x_2^1, x_2^2 \in V^{n-k}$ , then there exists a solution  $F : V^n \rightarrow W$  of equation (3.1) with

$$\|f(x) - F(x)\| \leq \frac{2}{|2|^{2n-k}} \sum_{j=1}^n \|x_j\|^t, \quad x \in V^n.$$

Similarly, Theorem 3.2 with the control function

$$\theta(x_1^1, x_1^2, x_2^1, x_2^2) = \prod_{i=1}^2 \prod_{j=1}^n \|x_{ji}\|^{t_{ji}}$$

gives the following outcome.

**Corollary 3.4.** *Assume that  $t_{ji} > 0$ , for  $i \in \{1, 2\}$  and  $j \in \{1, \dots, n\}$ , fulfill*

$$\sum_{i=1}^2 \sum_{j=1}^n t_{ji} > 2n - k,$$

$V$  is a normed space and  $W$  is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2 such that  $|2| < 1$ . If  $f : V^n \rightarrow W$  satisfies the inequality

$$\|\Psi(f)(x_1^1, x_1^2, x_2^1, x_2^2)\| \leq \prod_{i=1}^2 \prod_{j=1}^n \|x_{ji}\|^{t_{ji}}$$

for  $x_1^1, x_1^2 \in V^k$ ,  $x_2^1, x_2^2 \in V^{n-k}$ , then there exists a solution  $F : V^n \rightarrow W$  of equation (3.1) with

$$\|f(x) - F(x)\| \leq \frac{1}{|2|^{2n-k}} \prod_{j=1}^n \|x_j\|^{t_{j1}+t_{j2}}, \quad x \in V^n.$$

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